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Local Methods for Constructing Stationary Distribution Functions of Systems of Stochastic Differential Langevin-Type Equations: Noise Influence on Simple Bifurcation

I. I. Fedchenia[1]

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The influence is considered of two additive correlated noise effects on a two-dimensional quadratic-nonlinear system describing the behavior of two hydrodynamic modes. Using the method of Gaussian approximation, local characteristics of the distribution function are calculated, which are used to construct the global distribution function with the aid of the method of fraction-rational approximations. It is shown that for a system at whose bifurcation point the asymptotic stability is lost, in an expanded space of parameters (bifurcation parameter in the absence of noise plus noise parameters) there appears an instability zone within which the stationary distribution function does not exist. The effect of noise correlation on the stationary characteristics of the system is studied.

KEY WORDS: White noise; bifurcation; dynamical systems; hydrodynamic system; Gaussian approximation; functional-rational approximation; stationary distribution function.

1. INTRODUCTION. PERTURBATION AND LOCAL METHODS OF SOLVING THE FOKKER–PLANCK EQUATION

The local methods of ordinary differential equation analysis are powerful tools for constructing, for many general situations, a qualitative picture of the behavior of systems described by such equations.[1] Attempts to use these methods for stochastic Langevin-type equations meet with difficulties that limit the possibilities of such generalizations. Once white-noise-type fluctuations have been introduced into the system, any local problem turns

[1] Institute of Physics, BSSR Academy of Sciences, 220602, Minsk, USSR.

into a global one due to the presence in such fluctuations of arbitrarily high intensity bursts, which makes the system "feel" whatever remote boundaries. This feature of fluctuations leads to the appearance of transitions even between very distant equilibrium states and to their mutual influence. Practically all attempts to use the local method of describing distribution functions by the Fokker–Planck equation are based on the procedure that in quantum mechanics received the name "nonlinearization,"[2] consisting in representing the distribution function in the form $P = Ne^U$, where U is sought as Taylor expansions in the vicinity of given points (equilibrium states in the absence of noise, etc.) characteristic of the system. Although such "nonlinearization" has been used to seek wave functions since the late-1940s,[2] such representation was apparently first proposed for diffusion processes by Krasovskii.[3] He also analyzed the question of the convergence of Taylor expansions for U in some situations. The main difficulty consists in the fact that it is necessary to validate the procedure of cutting off the infinite chain of equations for the expansion coefficients U. This seems not to have been strictly done, and as is often the case, in applications one is orientated toward physical reasons for the obtained results. The cutoff of the infinite chain leads to the so-called "shifted" estimates for the true sought values. The results obtained by such a method have been analyzed in Ref. 4.

The investigation of systems near the instability threshold corresponding to the change of equilibrium states under the action of fluctuations has resulted in the concept of phase transitions induced by external noise. This effect was obtained theoretically for a number of exactly solved models[5–9] found experimentally[10–13] in a number of physical systems and consists in the fact that the equilibrium positions of the system in the absence of noise differ from the position of the stationary distribution function maxima which are interpreted as new equilibrium states; the noises deform the bifurcation surfaces in the parameter space passing through which the system changes it qualitative behavior (e.g., from the equilibrium position a limiting cycle is created, two equilibrium positions merge, and so on). If, in addition, we take into account the fact that when noise is introduced into the system the dimensionality of the parameter space increases by the number of parameters characterizing the noise, there appear new bifurcation surfaces passing through which the system goes to states which in the absence of noise did not exist at all. The local method has been used to solve these problems. Stratonovich[14] constructed the main term of local expansion of U for the general case of one equilibrium position and for several equilibrium positions in the presence of symmetry.

In the case of small fluctuations and for the two-dimensional systems of Langevin-type equations studied here [Eq. (1), Section 3] the computational scheme is as follows: representing the stationary distribution function to be sought in the form

$$P = N^{-1} \exp(-U_0/\varepsilon + U_1 + \varepsilon U_2 + \cdots)$$

and substituting it into the appropriate Fokker–Planck equation [Eq. (2), Section 3] and equating the terms with the same power of ε, we obtain the chain of partial differential equations in U_l, $l = 0, 1,...,$ (of first order this time). Write the first two:

$$N_{11}(U_0^{(1)})^2/2 + N_{12} U_0^{(1)} U_0^{(2)} + N_{22}(U_0^{(2)})^2/2 = U_0^{(1)} K_1 + U_0^{(2)} K_2 \qquad \text{(A)}$$

$$(N_{11} U_0^{(1)} + N_{12} U_0^{(2)} - K_1)\, U_1^{(1)} + (N_{12} U_0^{(1)} + N_{22} U_0^{(2)} - K_2)\, U_1^{(2)}$$
$$= K_1^{(1)} + K_2^{(2)} - (N_{11} U_0^{(11)} + 2N_{12} U_0^{(12)} + N_{22} U_0^{(22)})/2 \qquad \text{(B)}$$

Here we use the notation $\partial f/\partial x_i = f^{(i)}$, $i = 1, 2$.

In the work of Tolstopyatenko and Schimansky-Geier,[15] an additional "nonlinearization" procedure is proposed, which explicitly takes into account the probability density flux vorticity. This procedure, in principle, makes it possible to go beyond the scope of both the case of small fluctuations considered in Ref. 14 and the case of the presence of potentiality conditions where the stationary Fokker–Planck equation has an exact solution. The cutoff procedure of Ref. 15 was carried out on the principle of leaving a minimum number of Taylor expansion coefficients of the vorticity function required for the solvability of the corresponding equations for the expansion coefficients of U. The question of the bias of bifurcation points obtained from this theory has not been studied, but correlation has been made with the perturbation method of Ref. 14. It is shown that in this approximation the results coincide. The algebraic equations for Taylor coefficients of local expansion of the main asymptotic term U/ε do not contain any arbitrary parameters, which suggests that the approach of Ref. 15 is asymptotically equivalent to the approach of Ref. 14. On the other hand, from this it follows that using such a method, it is impossible, generally speaking, to satisfy any predetermined boundary conditions for P. It turns out that P found in such a manner for most important cases of steady states satisfies the so-called "natural" boundary conditions, i.e., it is equal to zero together with its derivatives. This just makes it possible to use the obtained solutions as physically adequate ones.

In the cases where it is necessary to satisfy boundary conditions other than "natural" ones, a different method should be used. Ventzel and Freidlin[16] created a strict mathematical theory of constructing distribution functions of diffusion processes with diffusion tensor elements uniformly tending to zero. The terms of expansion in powers of ε satisfy the equation in partial derivatives, but already of first order, and therefore the possibility exists of satisfying boundary conditions other than "natural" ones. This method, however, is essentially nonlocal (and therefore is extremely complicated from the computational point of view). In Refs. 17 and 18 a local method for constructing U was proposed. This method contains free parameters, by the choice of which it is possible to satisfy different boundary conditions or other relations superimposed on the sought solution (for example, it may be required that only the first n relations between the moments of random variables under consideration be fulfilled). This method explicitly takes into account the probability flow vorticity, as does the method of Ref. 15, but leads to recurrent chains of equations which may be solved sequentially. This method is appropriate in situations where a perturbed solution in the vicinity of the exact solution must be obtained. In the general case, the application of this method is impeded by the fact that in order to determine arbitrary constants which actually are local expansion coefficients of the probability flow vorticity, it is necessary to specify the corresponding local values, which are usually unknown. And attempts to associate these coefficients with the boundary values of the sought solution meet with great computational difficulties.

Even in the case of small fluctuations, which has received much study,[14,16] there exist many computational problems. As shown in a series of works by Graham and Tél,[19,20] the equations for the main asymptotic term of expansion $U_0/\varepsilon(A)$ (which are Hamilton–Jacobi equations), are integrable when and only when the drift and diffusion coefficients of the Fokker–Planck equation satisfy the conditions of potentiality.[21] Otherwise, U_0 may be nondifferentiable on certain manifolds. This is due to the absence of a sufficient number of first integrals in the canonical equations of motion for the characteristics of the corresponding Hamilton–Jacobi equations.

Another, more grave disadvantage of the method of U expansion in powers of ε (which is also called the ε-expansion method) is the fact that it is inapplicable in the vicinity of those points in the parameter space where the noise-free system loses its asymptotic stability. As a result, the external noise-induced phase transitions, which take place near the points of stability loss, cannot be described by such a method. The present work deals exactly with such a case.

The mathematical reason why the ε-expansion method is inapplicable in the vicinity of the point of asymptotic stability loss is the fact that the ε power series in the form of which U is sought is in reality a reciprocal power series of the bifurcation parameter μ such that the expansion term $U_n \sim \varepsilon^n/\mu^m$. Such a series converges quickly at large μ, i.e., far from the instability point, but begins to diverge somewhere in its vicinity.

At least for two-dimensional systems, it is possible to trace the source of nonanalyticity and relate it to the bifurcation type in the absence of noise. Let us do this. According to the ideas of catastrophe theory, the main qualitative peculiarities of the function are in the coefficients of its Taylor expansion of the lowest order. The equations for the expansion coefficients

$$U_{\left\{ {1 \atop 0} \right\}} = \sum_{i+j=0}^{i+j=\infty} U_{\left\{ {1 \atop 0} \right\}}^{\overbrace{11\ldots}^{i}\underbrace{22\ldots}_{j}} (x_1 - {}_0x_1)^i \, (x_2 - {}_0x_2)^j$$

are obtained by differentiating the left and right sides of (A) and (B) at the point $\{{}_0x_1, {}_0x_2\}$. If we choose one of the singularity points to be an expansion point $(K_1, K_2)\,|_{\{{}_0x_1, {}_0x_2\}} = 0$, then $U_0^1 = U_0^2 = 0$ immediately follows from (A), i.e., this point is a point of the U_0 extremum, and the systems of algebraic equations in $U_0^{11\ldots 22\ldots}$ are as follows:

$$(U_0^{ik})^{-1} K_j^k + K_i^k (U_0^{jk})^{-1} = N_{ij} \tag{C}$$

(summation over repeated indices is meant)

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & 2a_{11} + a_{22} & 2a_{12} & 0 \\ 0 & 2a_{21} & 2a_{22} + a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} U_0^{111} \\ U_0^{112} \\ U_0^{122} \\ U_0^{222} \end{pmatrix}$$

$$= \begin{pmatrix} K_1^{11} U_0^{11} + K_2^{11} U_0^{12} \\ K_1^{11} U_0^{12} + 2K_1^{12} U_0^{11} + 2K_2^{12} U_0^{12} + K_2^{11} U_0^{22} \\ K_1^{22} U_0^{11} + 2K_1^{12} U_0^{12} + 2K_2^{12} U_0^{22} + K_2^{22} U_0^{12} \\ K_1^{22} U_0^{12} + K_2^{22} U_0^{22} \end{pmatrix} \tag{D}$$

$$\cdots$$

$$\hat{A}_n^0 (U_0^{\underbrace{11\ldots}_{i}\underbrace{22\ldots}_{j}}) = f_n \qquad (i + j = n)$$

Taking into account the identity $N_{11} U_0^{11} + 2N_{12} U_0^{12} + N_{22} U_0^{22} = 2(K_1^1 + K_2^2)$ following from (C), the equations in

$$U_1^{\underbrace{11\ldots}_{i}\underbrace{22\ldots}_{j}}$$

are obtained:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} U_1^1 \\ U_1^2 \end{pmatrix} = \begin{pmatrix} K_1^{11} + K_2^{12} \\ K_1^{12} + K_2^{22} \end{pmatrix}$$
$$\qquad - \frac{1}{2} \begin{pmatrix} N_{11} U_0^{111} + 2N_{12} U_0^{112} + N_{22} U_0^{122} \\ N_{11} U_0^{112} + 2N_{12} U_0^{122} + N_{22} U_0^{222} \end{pmatrix} \tag{E}$$
$$\cdots$$
$$\hat{A}_n^1(U_1^{\underbrace{11\ldots}_{i}\,\underbrace{22\ldots}_{j}}) = f_n^1$$
$$\cdots$$

where $a_{ij} = U^{ik} N_{kj} - K_j^i$.

As is well known from catastrophe theory, the local form of the function at the extremum point undergoes qualitative changes when the determinant of the second-order derivative matrix changes sign. The form of the determinant for our case is very remarkable,

$$\det U_0^{ij} = 4(\mathrm{Tr}\, K_i^j)^2 \det K_i^j [(N_{11} K_2^1 - N_{22} K_2^1 - N_{12} K_1^1 + N_{12} K_2^2)^2$$
$$+ \det N_{ij} (\mathrm{Tr}\, K_i^j)^2]^{-1}$$

Since the numerator of this expression contains a value which is positive for any value of the parameters, $\det U_0^{ij}$ can change its sign when either $\mathrm{Tr}\, K_i^j$ or $\det K_i^j$ of the linear part of the matrix of the system under study in the absence of fluctuation changes its sign. It is known from the general theory of two-dimensional systems (see, for examqple, Ref. 30) that this exhausts all the types of bifurcations taking place with the linear part existing. This implies that when studying the influence of noise on bifurcation one should take into account at least one more term U_1 in the U expansion in terms of ε. If we could compute U_1, then the extremums of $U_0/\varepsilon + U_1$ would give approximate values of new locations of maxima and minima obtained from the conditions $(U_0/\varepsilon + U_1)^{(i)} = 0$, after which U could be reexpanded in the vicinity of the new maxima $({}_nx_1, {}_nx_2)$ and the determinant of the matrix of the second-order derivatives $U^{(ij)}({}_nx_1, {}_nx_2)$ computed. It would depend on N_{ij} in a nontrivial way and define new bifurcation surfaces in the expanded space of the noiseless bifurcation parameters plus noise parameters N_{ij}.

Let us see how this program may be realized. It is obvious from equations (E) for the coefficients of the U_1 linear part that in order to determine them it is necessary to know the coefficients U_0^{ijk} of the cubic terms of the U_0 expansion, which can be determined from system (D). To avoid awkward expressions for U_0^{ijk} and for subsequent coefficients we present only the expressions for the matrix determinants that should be other than zero for the solution of these systems to exist. Using equation (C), it is easy to obtain the following results:

$$\begin{aligned} a_{ij} &= U_0^{il} K_l^k (U_0^{jk})^{-1}, \qquad \det a_{ij} = \det K_j^i \\ \det A_3^0 &= \det K_j^i [\det K_j^i + 2\,\mathrm{Tr}(K_j^i)^2] \\ \det A_4^0 &= \mathrm{Tr}\, K_j^i \det K_j^i [3(\mathrm{Tr}\, K_j^i)^2 + 4 \det K_j^i] \\ \det A_2^1 &= \mathrm{Tr}\, K_j^i \det K_j^i \end{aligned} \tag{F}$$

For U_0, this means only one thing: in the vicinity of noiseless steady states at the bifurcation point the loss of analyticity takes place, since the corresponding partial differential equation (A) is quite solvable in this case. Of course, it may turn out that when calculating U_0^{ijk} according to the Kramer rule, $\det K_i^j$ cancels out in the numerator and denominator due to certain special properties (for example, symmetry) of the system under study. Then it is possible to determine these values and to compute U_1^i, U_2^{ij} from Eq. (E). It is seen, however, from Eq. (F) that $\det a_{ij} = \det K_j^i$, and therefore the solutions for U_1^i and U_2^{ij} do not exist either, for the same reason in the absence of certain special properties at the bifurcation point.

Nondifferentiability of U_0 in the vicinity of a certain manifold was first established by Ventzel and Freidlin[16] and studied by Graham and Tél[19,20] for a number of particular systems. It is possible, however, to go further and estimate the qualitative behavior of U_1. If we reject the expansion of U_0, U_1 into the Taylor series in favor of solving the partial differential equations (A), (B) by the method of characteristics, then, as is shown in Refs. 16, 19, and 20, the solution for U_0 can be obtained for all values of the parameters. The equation for characteristics for U_1 is of the form

$$dx_i/dt = N_{ik} U_0^{(k)} - K_i$$

the matrix of their linear part, as follows from (C) and the first equation of (F), being $U_0^{il} K_l^k (U_0^{jk})^{-1}$, whose determinant and trace both coincide with those of K_i^j. If we take into account that the solution of (B) is obtained by integrating the right-hand side, which is an increasing function of x_i, over characteristics (G), it becomes clear that the solution for U_1, which is bounded at $t \to \infty$, may be obtained only in the case when the trajectories of system (G) do not extend to infinity. As is well known,[30] however, singularity points can lose their stability for some bifurcation types and the system trajectories abandon their neighborhood at $t \to \infty$. For such types of bifurcation $U_1 \to \infty$ at $t \to \infty$, and it will be greater than U_0 at any small ε and at any point of the phase space. This means that for systems in which the noiseless bifurcation occurs passing through unstable states the

ε-expansion method can be used only when t is finite and cannot be used to obtain the stationary distribution function, since in this case the U/ε expansion into the ε power series is not even asymptotic. The Knobloch and Wiesenfeld approach,[31] based on the expansion of U in the vicinity of the central manifold, cannot be used in this case either, because the latter is nonexistent for such situations.

As is known from the general theory of functions,[22] this is an indication of either the nonexistance of the object [i.e., U, and therefore the absence of a nontrivial ($P \not\equiv 0$) stationary distribution function at $t \to \infty$], or/and the presence of ε and μ nonanalyticity points somewhere near (at small ε) the bifurcation point in the absence of noise. For the investigation of such cases the method of Gaussian approximation is adequate[23] when the sought function is considered to be locally Gaussian, the normal distribution parameters being found from the condition of fulfilling the required number of equations for moments. At small ε, using the Laplace method,[24] this approach may be extended to more general types of distributions and acquire the character of a calculation of the asymptotic expansion in a given small parameter $\delta(\varepsilon, \mu)$ which is a nonanalytical function of ε and μ. In practice, however, nonlinear equations even for second moments turn out to be very complicated for analysis, such that one is confined, as a rule, only to the Gaussian approximation. In Section 3, using the method of Gaussian approximation, local characteristics are calculated of the stationary distribution function of a two-dimensional dynamic system in the vicinity of the breakdown of one stable equilibrium position into two positions in the presence of fluctuations. It should be noted that for the cases of two equilibrium positions a global distribution function is constructed on the basis of the available data on equilibrium positions and the saddle point in the form of a fraction-rational approximation nondifferentiable on a given curve, which agrees with the results of Graham and Tél[19,20] about the nondifferentiability of U at $\varepsilon \to 0$.

2. THE PHYSICAL SYSTEM UNDER STUDY

In Sections 3 and 4 a problem taken from the hydrodynamics of vortex flows in ellipsoidal containments[25] is studied by the abovementioned methods with regard to fluctuations. Equations for v_0, v_1, and v_2 of dimensionless lower modes of the flow rate

$$\begin{aligned} \dot{v}_0 &= v_2^2 - v_1^2 - v_0 + R + f_0(t) \\ \dot{v}_1 &= v_0 v_1 - v_1 + f_1(t) \\ \dot{v}_2 &= -v_0 v_2 - v_2 + f_2(t) \end{aligned} \tag{I}$$

are obtained using the Galerkin procedure in the Helmholtz equation of an ideal incompressible fluid inside a unequiaxial ellipsoid. Here R is the analog of the Reynolds number, and f_0, f_1, and f_2 are δ-correlated fluctuation sources. The use of such a single-parametric three-mode model was justified by Gledzer *et al.*[25] and the field of application for the above simple model of fluctuation sources was discussed by Klyatskin.[26]

The results presented in the sections that follow are concerned with a two-dimensional particular case (I) where the component v_2 is not excited. In new variables more convenient for calculation the system takes on the form

$$\dot{x}_1 = M - x_1 - x_2^2 + \eta_{x_1}, \qquad \dot{x}_2 = x_1 \cdot x_2 + \eta_{x_2} \tag{II}$$

where $M = R - 1$, $x_1 = v_0 - 1$, and $x_2 = v_1$. Such a particular case corresponds to the Burgers model of the appearance of pulsations in the flow, where x_1 corresponds to the main flow and x_2 corresponds to pulsations. Equations (II) belong to the class of simple nonlinear quadratic systems in which bifurcations in the absence of noise have received a good deal of study (see, for example, Ref. 27).

For $M < 0$ in the absence of noise (II) has one asymptotic stable stationary solution $\{x_2 = 0,\ x_1 = M\}$, and for $M > 0$, two asymptotic stable stationary solutions $\{x_1 = 0,\ x_2 = \pm\sqrt{M}\}$; for $M = 0$, i.e., at the point of mergence, system (II) is unstable.

In Section 3, the local characteristics of the sought distribution function are calculated by the method of Gaussian approximation. These characteristics are used in Section 4 to construct the global distribution function by the method of fraction-rational approximations. Interpreting the stationary distribution function maxima as new steady stationary states, bifurcation surfaces are constructed in the expanded space of the parameters $\{M, N_{11}, N_{12}, N_{22}\}$ (where N_{11}, N_{12}, N_{22} are the intensities and correlation of noises η_{x_1}, η_{x_2}) at whose intersection the bistable and monostable regimes in (II) are changed by instability. The dependence is presented of the position of the distribution function maxima on the fluctuation powers and the correlation between them in each of these regions.

3. THE METHOD OF GAUSSIAN APPROXIMATION IN SINGULARLY PERTURBED FLUCTUATION PROBLEMS

In this section, the method of Gaussian approximation is used to seek the local characteristics of the two-dimensional system of Langevin equations

$$\dot{x}_1 = K_1(x_1, x_2) + \eta_1(t), \qquad \dot{x}_2 = K_2(x_1, x_2) + \eta_2(t) \tag{1}$$

where η_1 and η_2 are white noises in Stratonovich sense with

$$\langle \eta_i(t)\, \eta_j(t') \rangle = \varepsilon N_{ij} \delta(t'-t), \quad i, j = 1, 2, \qquad \text{at} \quad \varepsilon \to 0$$

Primary consideration will be given to the situation where the noise-free system (1) has one or two steady states, depending on the value of the bifurcation parameter. System (II) is the simplest system of such type (with lowest quadratic-type nonlinearity). At the same time it has all the main characteristic features of the problems with a nonunique equilibrium position. If the maxima of the stationary distribution function are considered as equilibrium states of a noisy system, the main goal of the local approach is to investigate the equilibrium position change depending on the powers, the noises introduced into the system, and the correlations between them, as well as to construct a new bifurcation diagram in the parameter space expanded at the cost of the parameters of fluctuation perturbations.

The Fokker–Planck equation for the stationary distribution function satisfying system (1) has the form

$$\frac{\varepsilon}{2}\left(N_{11}\frac{\partial^2 P}{\partial x_1^2} + 2N_{12}\frac{\partial^2 P}{\partial x_1\, \partial x_2} + N_{22}\frac{\partial^2 P}{\partial x_2^2}\right) = \frac{\partial}{\partial x_1}(K_1 P) + \frac{\partial}{\partial x_2}(K_2 P) \tag{2}$$

Let us seek the solution of (2) in the form $P = N \exp[-U(x_1, x_2)/\delta(\varepsilon, M)]$, where $\delta(0, M) = 0$ is a given unknown function (generally speaking, a nonanalytical one) of ε and of the bifurcation parameter M of the problem of (II). Then, at small ε, according to the Laplace method,[24] the main contribution to any integrals $\int P(x_1, x_2)\, \varphi(x_1, x_2)\, dx_1\, dx_2$ will be made by the vicinity of the minimum U and, in the case of the presence of an x_1, x_2-nondegenerate quadratic part in U, the fundamental term asymptotic at $\varepsilon \to 0$ will be determined by this part only. The contribution of higher degrees of the U expansion in x_1, x_2 will be made by the following terms of the asymptotic expansion in δ. Confining ourselves further to the main asymptotic terms, we shall consider U as an x_1, x_2-quadratic form only. This corresponds to the so-called Gaussian approximation.[23] Thus, in this case

$$P = \frac{1}{2\pi(\det \hat{M})^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \hat{M}^{-1}(\mathbf{x}-\mathbf{m})\right] \tag{3}$$

where

$$\mathbf{m} = \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix}, \qquad \hat{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{pmatrix}$$

are, respectively, the mean vector and the dispersion matrix. To determine them, five equations are required. Let us obtain them by substituting (3) into (2) and integrating with prior multiplication by the corresponding factors x_i or $x_i x_j$,

$$\langle K_i \rangle = 0 \tag{4}$$

$$\langle K_i x_j \rangle + \langle K_j x_i \rangle + N_{ij} = 0 \tag{5}$$

Here $i = 1, 2$ and

$$\langle \cdots \rangle = \int_{R^2} P(x_1, x_2)(\cdots)\, dx_1\, dx_2$$

How should one understand Eqs. (4) and (5) in the case where the system has several equilibrium positions? In this case the sought distribution function may be represented locally in the form of (3). Then the means in Eqs. (4) and (5) should be treated as conditional means in the regions of maxima. However, due to the presence of the small parameter ε, the main contribution to the main asymptotics of mean values, according to the Laplace method,[24] looks as if integration were made over the entire plane R^2. Writing Eqs. (4) and (5) separately for each maximum and solving them, we obtain local values $\mathbf{m}$ and $\hat{M}$. Then the local information thus obtained may be used to construct approximations of the global distribution function. This will be done below with the aid of fraction-rational approximations.

We now turn to the analysis of system (II). Let us begin with the case of one equilibrium position in the absence of noise: $M < 0$.

In the system of coordinates where the noise-free equilibrium position is zero, the equations (II) take on the form

$$\dot{x} = -x - y^2 + \eta_1, \qquad \dot{y} = My + xy + \eta_2 \tag{6}$$

where $x = x_1 - M$ and $y = x_2$.

In this case the equalities for the moments of (4), (5) are of the form

$$\begin{aligned} m_1 + (m_2)^2 + M_{22} &= 0 \\ M \cdot m_2 + m_1 \cdot m_2 + M_{12} &= 0 \end{aligned} \tag{7}$$

$$\begin{aligned} &(m_1)^2 + M_{11} + m_1 \cdot M_{22} + m_2 \cdot M_{12} + m_2(m_1 \cdot m_2 + M_{12}) - \varepsilon N_{11}/2 = 0 \\ &(M-1)(m_1 \cdot m_2 + M_{12}) + m_2[(m_1)^2 + M_{11}] - m_2[(m_2)^2 + M_{22}] \\ &\qquad - 2m_2 \cdot M_{22} + 2m_1 \cdot M_{12} + \varepsilon N_{12} = 0 \\ &M[(m_2)^2 + M_{22}] + m_1[(m_2)^2 + M_{22}] + 2m_2 \cdot M_{12} + \varepsilon N_{22}/2 = 0 \end{aligned} \tag{8}$$

Here the known expressions for the Gaussian means are used:

$$\langle x^2\rangle=(m_1)^2+M_{11};\qquad \langle xy\rangle=m_1\cdot m_2+M_{12};\qquad \langle y^2\rangle=(m_2)^2+M_{22}$$
$$\langle x^3\rangle=(m_1)^3+3m_1\cdot M_{11};\qquad \langle x^2y\rangle=m_2\cdot M_{11}+2m_1\cdot M_{12}+(m_1)^2\cdot m_2$$
$$\langle xy^2\rangle=m_1\cdot M_{22}+2m_2\cdot M_{12}+m_1\cdot(m_2)^2;\qquad \langle y^3\rangle=(m_2)^3+3m_2\cdot M_{22} \tag{9}$$

The matrix $\hat{M}$ elements may easily be expressed in terms of the vector elements $\mathbf{m}$:

$$\begin{aligned} M_{11}&=-M\cdot m_1+\varepsilon(N_{11}+N_{22})/2\\ M_{12}&=-(M+m_1)\cdot m_2\\ M_{22}&=-[m_1+(m_2)^2] \end{aligned} \tag{10}$$

which must satisfy the system of equations

$$\begin{aligned} &2(m_2)^3+[M(1-M)+\varepsilon(N_{11}+N_{22})/2+3(1-M)\,m_1\\ &\qquad -2(m_1)^2]\,m_2+\varepsilon N_{12}=0\\ &(m_1)^2+2[M+(m_2)^2]\,m_1+2M(m_2)^2-\varepsilon N_{22}=0 \end{aligned} \tag{11}$$

System (11) is of fifth order and cannot be solved in radicals. However, the sought roots may be found with the aid of the fixed-point theorem. Indeed, of all the roots of system (11), we need only the one that tends to zero at $\varepsilon\to 0$, which corresponds to the tendency of the maximum position to the equilibrium position of the noise-free system, and the tendency of $\hat{M}$ (10) to zero provides the tendency of p to the δ-function. Again, if we express the sought root m_2 by the Cardano formula in terms of the coefficients of the first equation of (11),

$$m_2=-2(-p/3)^{1/2}\cos(\alpha/3+\pi/3) \tag{12}$$

where

$$p=[M(1-M)+\varepsilon(N_{11}+N_{22})/2+3(1-M)\,m_1-2(m_1)^2]/2$$

and $\cos\alpha=-\varepsilon N_{12}/4[-(p/3)^3]^{1/2}$, and express m_1 by the known formula in terms of the coefficients of the second equation,

$$m_1=-[M+(m_2)^2]-\{[M+(m_2)^2]^2+\varepsilon N_{22}-2M(m_2)^2\}^{1/2} \tag{13}$$

and substitute (12) into (13) and (13) into (12), the problem of finding the sought root will be reduced to finding the fixed point of mapping

$m_1 = f(m_1)$ and $m_2 = \varphi(m_2)$. Estimations and numerical calculations show that the derivative of the functions f and φ at small M and ε are small and therefore the iterations of the above mappings beginning with zero values of m_1 and m_2 quickly converge. Practically the first iteration brings us to the region of the root. Taking this into account, we obtain for the first approximation of m_1 and m_2

$$\begin{aligned} {}_1m_1 &= -[M + (\varepsilon N_{12}/2)^{2/3}] \\ &\quad - \{[M + (\varepsilon N_{12}/2)^{2/3}]^2 + \varepsilon N_{22} - 2M(\varepsilon N_{12}/2)^{2/3}\}^{1/2} \\ {}_1m_2 &= -(\varepsilon N_{12}/2)^{1/3} \end{aligned} \tag{14}$$

It can easily be seen from (14) that ${}_1m_1$ and ${}_1m_2$ really tend to zero at $\varepsilon \to 0$, but they are not ε-analytical in the vicinity of $\varepsilon = 0$. [In general, neither is their behavior described by power functions with rational indexes whose presence in the expressions of (14) is the result of the approximate procedure of seeking the roots.]

The condition of the existence of a nontrivial stationary solution is the condition of the positivity of the dispersion matrix determinant, whose elements are given by expressions (10):

$$M_{11}M_{22} - (M_{12})^2 > 0 \tag{15}$$

Figure 1 shows the family of curves separating the regions of positivity and negativity of Det $\hat{M}$ in the $\{\varepsilon, M\}$ parameter space for different relations between N_{11}, N_{22}, and N_{12}. It is seen from this family of curves that the main qualitative effect of fluctuations is the appearance of an instability region instead of one point $M = 0$, i.e., at any small ε there exists an M_{cr} such that at $|M| < |M_{cr}|$ the nontrivial state is absent from the system.

We now turn to the case of $M > 0$. In the absence of noises for these values of M there are two asymptotically stable equilibrium positions with coordinates $\{x_1 = 0, \ x_2 = \sqrt{M}\}$, $\{x_1 = 0, \ x_2 = -\sqrt{M}\}$. Let us first calculate the Gaussian approximation parameters for each equilibrium position. For this purpose we rewrite Eq. (11) in coordinates in which the noise-free equilibrium state is at zero. It is sufficient to do this only for one equilibrium position, since by virtue of the existing symmetry in the equations of (11) all the expressions for the second equilibrium position are obtained simply by changing the sign at $\sqrt{M}$ and N_{12}. Thus, in the coordinates $x_1 = x$, $x_2 = y + \sqrt{M}$, Eqs. (II) assume the form

$$\dot{x} = -x - 2\sqrt{M}\,y - y^2 + \eta_1, \qquad \dot{y} = \sqrt{M}\,x + xy + \eta_2 \tag{16}$$

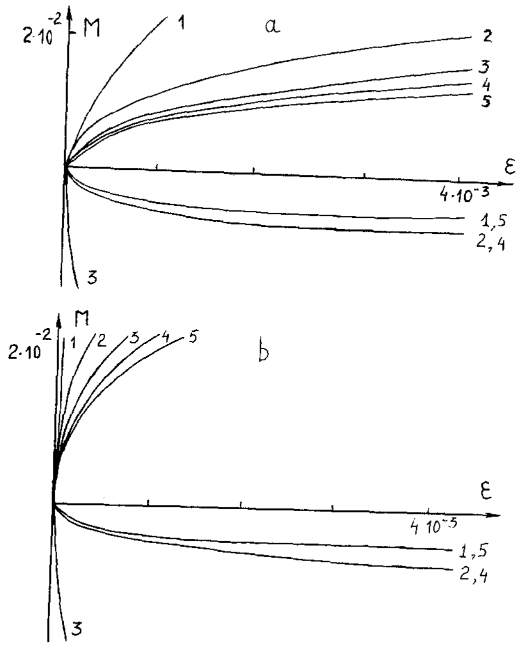

Fig. 1. Bifurcation diagram in the $\{\varepsilon, M\}$ parameter space for the right-hand maximum, for different values of the N_{11}/N_{22} ratio: (a) 0.26, (b) 1.31; at different values of correlation coefficient R: (1) -0.9, (2) -0.5, (3) 0, (4) 0.5, (5) 0.9. For the left-hand maximum the sign before R must be reversed. The instability zone is inside the curves.

With the use of (9), Eqs. (4), (5) give

$$\begin{aligned} M_{11} &= Mm_1 + \varepsilon(N_{11} + N_{22})/2 \\ M_{12} &= -\sqrt{M}\, m_1 - m_1 m_2 \\ M_{22} &= -m_1 - 2\sqrt{M}\, m_2 - (m_2)^2 \end{aligned} \tag{17}$$

For m_1 and m_2 we obtain the system of nonlinear equations

$$\begin{gathered} m_1^2 + [M + 4\sqrt{M}\, m_2 + (m_2)^2]\, m_1 - \varepsilon N_{22}/2 = 0 \\ (m_2)^3 + 3\sqrt{M}\, (m_2)^2 + [2M + \varepsilon(N_{11} + N_{22})/4 + m_1(M+3)/2 - (m_1)^2]\, m_2 \\ + \sqrt{M}\, [\varepsilon(N_{11} + N_{22})/4 + (M+3)m_1/2 - (m_1)^2] + \varepsilon N_{12}/2 = 0 \end{gathered} \tag{18}$$

Proceeding as in the case of $M < 0$, we reduce the problem of calculating the roots of system (18) tending to zero at $\varepsilon \to 0$ to the problem of calculating the fixed points of the mappings

$$m_1 = \frac{M + 4\sqrt{M}\, m_2 + (m_2)^2}{2} - \frac{\{[M + \sqrt{M}\, m_2 + (m_2)^2]^2 + 2\varepsilon N_{22}\}^{1/2}}{2} \tag{19}$$

$$m_2 = -\sqrt{M} + 2\sqrt{p/3}\cos(\alpha/3) \tag{20}$$

where

$$p = M - \varepsilon(N_{11} + N_{22})/4 - m_1(M+3)/2 + (m_1)^2$$
$$\cos\alpha = -\varepsilon N_{12}/4(p/3)^{3/2}$$

As in the case of $M < 0$, it is verified numerically that the derivatives of mappings (19) and (20) at small ε and M are very small, which permits us to write as a first approximation

$$_1m_1 = \frac{M + 4\sqrt{M}(_0m_2) + (_0m_2)^2}{2} - \frac{\{[M + \sqrt{M}(_0m_2) + (_0m_2)^2]^2 + 2\varepsilon N_{22}\}^{1/2}}{2}$$

$$_1m_2 = -\sqrt{M} + 2[(_0p)/3]^{1/2}\cos(\alpha/3) \tag{21}$$

where

$$_0m_1 = -M/2 + (M^2 + 2\varepsilon N_{22})^{1/2}/2$$
$$_0m_2 = -\sqrt{M} + 2[(_0p)/3]^{1/2}\cos(_0\alpha/3)$$
$$_0p = M - \varepsilon(N_{11} + N_{22}/4) - (M+3)(_0m_1)/2 + (_0m_1)^2$$
$$\cos(_0\alpha) = -\varepsilon N_{12}/4(_0p/3)^{3/2}$$

The condition of (15) is in this case the condition of the existence of one of the maxima whose local characteristic is described by expressions (17), (21). As in the case of $M < 0$, there appears an instability zone within which a stationary solution does not exist. Note, however, an interesting fact: the presence of correlation ($N_{12} \neq 0$) breaks the system symmetry such that if at $N_{12} = 0$ the right-hand and left-hand maxima disappear simultaneously when passing through the curve separating the stability and instability zones, at $N_{12} \neq 0$ there exist such points in the space of the parameters $\{M, \varepsilon\}$ that when, say, the left-hand maximum has disappeared while the right-hand one remains, this position is reversed as the sign of N_{12} changes (Fig. 1).

Besides the positions of the maxima, a very important characteristic point of the stationary distribution functions with a nonunique maximum is the saddle point located at low noise near the unstable equilibrium position. The local characteristics of the distribution function at this point are important in calculating the times of being in one of the equilibrium positions,[28] since at low noise, the system makes, with most probability, transitions through the vicinity of this point. Unfortunately, we cannot make use of the methods of Gaussian approximation to calculate the local characteristics of the saddle point, as done above for the investigation of the maxima, because the use of the Laplace method is restricted to functions of the form of (3) with $\det \hat{M} > 0$. And the saddle point is of saddle character ($\det \hat{M} < 0$), and the infinite-limit integrals, to whose calculation the problem in the Laplace method is reduced, diverge.

However, one may use the circumstance that to obtain the Gaussian approximation parameters of (3) one may use not only x_i and $x_i x_j$, but any functions [if only there exist integrals in Eqs. (4) and (5)]. Let us assume that in the vicinity of the noise-free unstable equilibrium position the distribution function is of the Gaussian form $c \exp[-(\mathbf{x}-\mathbf{s})^T \hat{S}^{-1}(\mathbf{x}-\mathbf{s})]$, where the quadratic form in the exponent index is not, however, sign-defined and, as is known, it has the saddle form. Using linear orthogonal substitution of variables

$$\mathbf{x} = \hat{T}\mathbf{y} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

we may reduce this form to $(y_1 - m_1)^2/2M_{11} - (y_2 - m_2)^2/2M_{22}$. Then, for the functions defining M_{11}, M_{22}, m_1, m_2, and s and c we choose the functions $y_1 \exp[-(y_1 - m_1)^2/M_{11}]$ and $y_i y_j \exp[-(y_1 - m_1)^2/M_{11}]$. For such functions all the integrals in expressions (4), (5), as may easily be seen, will already exist and represent Gaussian means with the function $\exp[-(y_1 - m_1)^2/2M_{11} - (y_2 - m_2)^2/2M_{22}]$. Performing all the operations of averaging the Fokker–Planck equation (2) for such functions with the distribution function whose local form is given above, we obtain for Eqs. (11) in coordinates (6):

$$-3\tilde{K}_{11} m_1 + \tilde{K}_{12} m_2 + cM_{22} - s m_1 m_2 + c(m_2)^2 + \varepsilon m_1 \tilde{N}_{11}/M_{11} = 0 \tag{22}$$

$$\begin{aligned} &-\tilde{K}_{12} m_1 + (\tilde{K}_{22} - 2\tilde{K}_{11})\, m_2 - 2sM_{22} - sM_{11} - s(m_1)^2 - 2s(m_2)^2 \\ &\quad + c m_1 m_2 + \varepsilon m_2 \tilde{N}_{11}/M_{11} = 0 \end{aligned} \tag{23}$$

$$\begin{aligned} &-4\tilde{K}_{11} M_{11} - 4s m_2 M_{22} - 4\tilde{K}_{11} m_1^2 + 2\tilde{K}_{12} m_1 \cdot m_2 - 4s m_1^2 m_2 \\ &\quad + 2c m_1 (M_{22} + m_2^2) + \varepsilon \tilde{N}_{11} m_1^2/M_{11} + 2\varepsilon \tilde{N}_{11} = 0 \end{aligned} \tag{24}$$

$$\tilde{K}_{12}[M_{22}+(m_2)^2]-\tilde{K}_{12}[M_{11}+(m_1)^2]+(\tilde{K}_{22}-3\tilde{K}_{11})\,m_1 m_2$$
$$-3sm_1[M_{22}+(m_2)^2]+m_2 c[3M_{22}+(m_2)^2]$$
$$+(cm_2-sm_1)[3M_{11}+(m_1)^2]+\varepsilon\tilde{N}_{11}m_1 m_2/M_{11}-\varepsilon\tilde{N}_{12}=0 \quad (25)$$
$$-2(\tilde{K}_{11}-\tilde{K}_{22})\,M_{22}-2\tilde{K}_{12}m_1 m_2-2(\tilde{K}_{11}-\tilde{K}_{22})(m_2)^2-6sm_2 M_{22}$$
$$+cm_1 M_{22}-sM_{11}m_2+cm_1(m_2)^2-s(m_1)^2 m_2-2s(m_2)^3$$
$$+\varepsilon\tilde{N}_{11}(m_2)^2/M_{11}+\varepsilon\tilde{N}_{11}M_{22}/M_{11}+\varepsilon N_{22}=0 \quad (26)$$

where

$$\tilde{K}=\begin{pmatrix} Ms^2-c^2 & -(M+1)\,cs \\ -(M+1)\,cs & Mc^2-s^2 \end{pmatrix}$$

is the matrix of the linear portion of the Fokker–Planck equation drift vector, and

$$\tilde{N}=\varepsilon\begin{pmatrix} N_{11}c^2-2csN_{12}+N_{22}s^2 & N_{11}cs+N_{12}(c^2-s^2)-N_{22}cs \\ N_{11}cs+N_{12}(c^2-s^2)-N_{22}cs & N_{11}s^2+2csN_{12}+N_{22}c^2 \end{pmatrix}$$

is the diffusion matrix on orthogonal transformation of the coordinates $\hat{T}$. The solution of this very cumbersome system can be obtained at small ε by the iteration method used to solve the systems of (11) and (18). Let us seek a solution of the system of equations (22)–(26) for which $\lim_{\varepsilon\to 0} M_{11}$, M_{22}, m_1, $m_2=0$ and $\lim_{\varepsilon\to 0} s=1$ and therefore $\lim_{\varepsilon\to 0} c=0$. This solution will locally describe a distribution function whose saddle point tends in the absence of noise to the unstable equilibrium position, and the transformation matrix $\hat{T}$ changes to the matrix of the

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

orthogonal rotation through $\pi/2$, which corresponds to the form of the saddle phase picture of the noise-free system (6).[25] [One can immediately make sure that the limiting values of the sought solutions satisfy, in the limit of $\varepsilon\to 0$, the system of (22)–(26).]

Choosing from (24) ${}_0s=1$, ${}_0c=0$ as a zero approximation, we obtain

$$({}_0M_{11})=\varepsilon N_{22}/2(M+{}_0m_2) \quad (27)$$

Here, $({}_0m_1)=0$ has been chosen. It will be seen below that this value is of the next order of ε infinitesimal. We obtain from (26)

$$({}_0M_{22})=\varepsilon\,\frac{2N_{11}(M+{}_0m_2)-{}_0m_2 N_{22}}{4(1+2\,{}_0m_2)(M+{}_0m_2)} \quad (28)$$

Here the values of $({}_0m_2)^2$ and $({}_0m_2)^3$ have been rejected as small compared to the remaining ones.

Substituting (27)–(28) into (23), we obtain the equation for ${}_0m_2$:

$$4({}_0m_2)^3 + 2(1+2M)({}_0m_2)^2 + [2M + \varepsilon(N_{11}+N_{22})]({}_0m_2) + 2\varepsilon M N_{11} + \varepsilon N_{22} = 0 \tag{29}$$

the root tending to zero at $\varepsilon \to 0$ being taken as the solution:

$${}_0m_2 = -\frac{1+2M}{6} - 2\sqrt{p}\cos\left(\frac{\alpha}{3} - \frac{\pi}{3}\right)$$

where

$$p = \frac{1}{3\cdot 4}\left[\frac{1-2M+4M^2}{3} - \varepsilon(N_{11}+N_{22})\right]$$

$$\cos\alpha = -\frac{q}{2(p)^{3/2}}$$

$$q = \frac{1}{12}(1+2M)\left[\frac{1-5M+4M^2}{9} - \frac{\varepsilon(N_{11}+N_{22})}{2}\right] + \varepsilon\frac{2MN_{11}+N_{22}}{4}$$

The first approximation for c is found from Eq. (25),

$${}_1c = -\varepsilon N_{12}/\{(1+M)[{}_0M_{22} - {}_0M_{11} + ({}_0m_2)^2] + {}_0m_2[3({}_0M_{22} + {}_0M_{11}) + ({}_0m_2)^2]\} \tag{30}$$

It is seen from (30) that ${}_0c$ is close to zero at small N_{12} (weak correlation noises) and at small N_{22}. Due to the presence in the denominator of the difference ${}_0M_{22} - {}_0M_{11}$ for another combination of parameters, it would be better to choose another zero approximation for s and c. Now we find from Eq. (22) the first approximation for m_1,

$${}_1m_1 = {}_1c[{}_0M_{22} + ({}_0m_2)^2 + (1+M)\,{}_0m_2]/(M - {}_0m_2) \tag{31}$$

It is seen from (29) for ${}_0m_2$ that always ${}_0m_2 < 0$, so the denominator in (31) never becomes zero. Repeating the above procedure, we can find further approximations of the sought roots. The values of the dispersion matrix and the mean vector in the initial variables of (6) are obtained by using the inverse transformation $\hat{T}^{-1}$, whose parameters as a first approximation can be found:

$$\mathbf{S} = \begin{pmatrix} S_x \\ S_y \end{pmatrix} = \hat{T} \begin{pmatrix} {}_1m_1 \\ {}_0m_2 \end{pmatrix}$$
$$\hat{S} = \begin{pmatrix} S_{xx} & S_{xy} \\ S_{xy} & S_{yy} \end{pmatrix} = \hat{T} \begin{pmatrix} {}_0M_{11} & 0 \\ 0 & {}_0M_{22} \end{pmatrix} \hat{T}^{-1} \tag{32}$$

where

$$\hat{T} = \begin{pmatrix} {}_1c & 1 \\ -1 & {}_1c \end{pmatrix}$$

4. CONSTRUCTION OF THE GLOBAL DISTRIBUTION FUNCTION BY MEANS OF FRACTION-RATIONAL APPROXIMATION

The local characteristics calculated above provide information on the influence of noise upon the equilibrium position coordinates and on the maximum half-widths. However, the local values do not provide such important information as the relative (relative to one another) value of maxima. Without knowing them, it is impossible to determine the time during which the system is in the vicinity of one of the equilibrium positions,[28] which is a very important characteristic of bistable systems as used in practice.

Let us seek the global distribution function in the form Ne^U, choosing U in the form of a fraction-rational function such that its first and second derivatives at the maximum and minimum points coincide with the corresponding values obtained by the local analysis. The additional requirement placed on the form of the function U is its positive-definiteness at infinity. To correlate the absolute values of U at extreme points, which cannot be found from the local analysis, we use the values of the barrier heights between two wells of U, which are assumed to be approximately equal to the difference of the Gaussian approximation values at the saddle point and at the point of each maximum. We also use the fact that an arbitrary constant may be added to the functon, so that the absolute height of one of the extreme points may be fixed arbitrarily.

The following designations are used: x_0^i denotes the saddle point coordinates (32); x_R^i denotes the right-hand maximum coordinates (21); x_L^i denotes the left-hand maximum coordinates [the same formula (21) in which $\sqrt{M}$ is replaced by $-\sqrt{M}$]; U_0^{ij} is the matrix inverse to the matrix of momenta ($U = \hat{M}^{-1}$) at the saddle point (32); U_R^{ij} is the matrix inverse to the matrix of momenta at the right maximum point; and U_L^{ij} is the matrix inverse to the matrix of momenta at the left-hand maximum point [the same formula (17) in which $\sqrt{M}$ is replaced by $-\sqrt{M}$].

Let us seek the global distribution function in the form Ne^U, where U is of the form

$$U=\begin{cases} U_R=\left[1-\dfrac{1}{\phi_R(x-x_0)+1}\right][C_R+\psi_R(x-x_R)] & \text{to the right of } x_0 \\ U_L=\left[1-\dfrac{1}{\phi_L(x-x_0)+1}\right][C_L+\psi_L(x-x_L)] & \text{to the left of } x_0 \end{cases} \tag{33}$$

The functions $\phi_{R,L}$, $\psi_{R,L}$, and the constants $C_{R,L}$ are determined from the conditions of coincidence of the first- and second-order derivatives with the corresponding local approximations P at the points x_0, x_L, and x_R, as well as from the condition of U continuity at the point x_0. The functions $\phi_{R,L}$ and $\psi_{R,L}$ are chosen in the form of x^i second-order homogeneous polynomials. Then, $\phi_{R,L}(x_0)=0$ and $\psi_R(x_R)=\psi_L(x_L)=0$, with the form of U being chosen such that $U_R(x_0)=U_L(x_0)=0$. Here we have made use of the fact that U is determined up to the additive constant. The function U is a continuous function twice differentiable at the point x_0 with a break of the first and higher derivatives on the curve determined by the equation $U_R(x)=U_L(x)$. At low noise, as follows from the results of Ventzel and Freidlin[16] and Graham and Tél,[19,20] the true distribution function tends to a function nondifferentiable on a given curve. Therefore, for the given case, (33) will be a good approximation. As to the study of phase transitions induced by external noise, the main information on the maxima shift and the saddle point as well as on how the bifurcation occurs (through the instability region in the parameter space) has already been obtained by the local method. Of greatest interest in constructing the global distribution function is finding the constants C_R and C_L, which determine the relative height of the maxima and therefore the time during which the system is present in the vicinity of each of them.

Below, calculations are given for finding the parameters of U_R. The corresponding expressions for U_L are obtained by substituting R for L.

Let us introduce the height of the "potential well" Δ_R equal to

$$\begin{aligned}\Delta_R&=U_R(x_0)-U_R(x_R)=U_R(x_0)\\ &=\tfrac{1}{2}U_R^{11}(x_0^1-x_R^1)^2+U_R^{12}(x_0^1-x_R^1)(x_0^2-x_R^2)+\tfrac{1}{2}U_R^{22}(x_0^2-x_R^2)^2\end{aligned} \tag{34}$$

In terms of the parameters of (33), this value is equal to

$$\Delta_R=-\frac{\phi_R(x_R-x_0)}{\phi_R(x_R-x_0)+1}C_R \tag{35}$$

The purpose of further manipulations is to obtain an algebraic equation for $\phi_R(x_R - x_0)$ in terms of which all the coefficients of the quadratic forms ϕ_R and ψ_R and of C_R are expressed. Since x_0 is the extreme point, $\phi_R^1 = \phi_R^2 = 0$. Putting now the partial second-order derivatives of U_R given by expression (33) at the point x_0 equal to the matrix elements of the moments for the saddle point calculated by the local method, we obtain

$$\phi_R^{ij} = \frac{U_0^{ij}}{C_R + \psi_R(x_0 - x_R)} \qquad (i, j = 1, 2) \tag{36}$$

Multiplying (36) by $(x_R^i - x_0^i)(x_R^j - x_0^j)$ and adding, we arrive at a chain of equalities

$$\begin{aligned}
&\frac{1}{2}\phi_R^{11}(x_R^1 - x_0^1)^2 + \phi_R^{12}(x_R^1 - x_0^1)(x_R^2 - x_0^2) + \frac{1}{2}\phi_R^{22}(x_R^2 - x_0^2)^2 \\
&\quad = \phi_R(x_R - x_0) \\
&\quad = \frac{\frac{1}{2}U_0^{11}(x_R^1 - x_0^1)^2 + U_0^{12}(x_R^1 - x_0^1)(x_R^2 - x_0^2) + \frac{1}{2}U_0^{22}(x_R^2 - x_0^2)^2}{C_R + \psi_R(x_0 - x_R)} \\
&\quad = \frac{U_0(x_R)}{C_R + \psi_R(x_0 - x_R)} \\
&\quad = \frac{\Delta_0}{C_R + \psi_R(x_0 - x_R)}
\end{aligned} \tag{37}$$

whence

$$\psi_R(x_0 - x_R) = \Delta_0/\phi(x_R - x_0) - C_R \tag{38}$$

whereupon (36) is expressed in terms of $\phi_R(x_R - x_0)$

$$\phi_R^{ij} = U_0^{ij}\phi_R(x_R - x_0)/\Delta_0 \tag{39}$$

We now turn to the calculation of the quadratic form ψ_R parameters. From the condition that x_R is the extreme point, we find

$$\psi_R^i = -\frac{(\partial/\partial x_i)\,\phi_R(x_R)}{\phi_R(x_R - x_0)}\, C_R = -C_R\, \frac{\phi_R^{ii}(x_R^i - x_0^i) + \phi_R^{ij}(x_R^j - x_0^j)}{\phi_R(x_R - x_0)} \tag{40}$$

Putting now the partial second-order derivatives of U_R given by expression (33) at the point x_R equal to the inverse matrix elements of moments for the right maximum point x_R, we obtain

$$U_R^{ij}=\left[\frac{\phi_R^{ij}}{[\phi_R(x_R-x_0)+1]^2}-2\,\frac{(\partial/\partial x_i)\,\phi_R(x_R-x_0)(\partial/\partial x_j)\,\phi_R(x_R-x_0)}{[\phi_R(x_R-x_0)+1]^3}\right]C_R$$
$$+\frac{[\partial\phi_R(x_R-x_0)/\partial x_i]\,\psi_R^j+[\partial\phi_R(x_R-x_0)/\partial x_j]\,\psi_R^i}{[\phi_R(x_R-x_0)+1]^2}$$
$$+\frac{\phi_R(x_R-x_0)}{\phi_R(x_R-x_0)+1}\,\psi_R^{ij} \tag{41}$$

whence ψ_R^{ij} may be determined. Multiplying (41) by $(x_0^i-x_R^i)(x_0^j-x_R^j)$ and adding, we arrive at a chain of equalities

$$\frac{1}{2}\,U_R^{11}(x_0^1-x_R^1)^2+U_R^{12}(x_0^1-x_R^1)(x_0^2-x_R^2)+\frac{1}{2}\,U_R^{22}(x_0^2-x_R^2)^2$$
$$=U_R(x_0)=\Delta_R$$
$$=\left[\frac{\phi_R(x_R-x_0)}{[\phi_R(x_R-x_0)+1]^2}-4\,\frac{\phi_R^2(x_R-x_0)}{[\phi_R(x_R-x_0)+1]^3}\right]$$
$$-\frac{2\phi_R(x_R-x_0)[(\partial/\partial x_1)\,\psi_R(x_0^1-x_R^1)+(\partial/\partial x_2)\,\psi_R(x_0^2-x_R^2)]}{[\phi_R(x_R-x_0)+1]^2}$$
$$+\frac{\phi_R(x_R-x_0)}{\phi_R(x_R-x_0)+1}\left[\psi_R(x_0-x_R)-\frac{\partial}{\partial x_1}\,\psi_R(x_0^1-x_R^1)\right.$$
$$\left.-\frac{\partial}{\partial x_2}\,\psi_R(x_0^2-x_R^2)-C_R\right] \tag{42}$$

Using the identity following from (40),

$$\frac{\partial}{\partial x_1}\,\psi_R(x_0^1-x_R^1)+\frac{\partial}{\partial x_2}\,\psi_R(x_0^2-x_R^2)=2C_R \tag{43}$$

formula (38) connecting $\psi_R(x_0-x_R)$ and $\phi_R(x_R-x_0)$, and formula (35) connecting Δ_R and C_R, we arrive at the equation for $\phi_R(x_R-x_0)$:

$$\Delta_R\phi_R^2(x_R-x_0)+(13\Delta_R+\Delta_0)\,\phi_R(x_R-x_0)+6\Delta_R+\Delta_0=0 \tag{44}$$

The condition

$$\lim_{x_R\to x_0}\phi_R(x_R-x_0)=0$$

and the fact that $\Delta_R<0$ make it possible to choose the desired root:

$$\phi_R(x_R-x_0)=\frac{-13\Delta_R-\Delta_0-(97\Delta_R^2+14\Delta_0\Delta_R+\Delta_0^2)^{1/2}}{6\Delta_R} \tag{45}$$

Now all the sought values may be expressed by means of $\phi_R(x_R - x_0)$ given by formula (45) in terms of the already calculated local values, which in turn are expressed in terms of the initial parameters of the problem.

The general view of the sought distribution function is given in Fig. 2. Note that by virtue of the initial symmetry in the problem at $N_{12} = 0$, $C_R/C_L = 1$, both peaks of the distribution function are completely symmetrical. At $N_{12} \neq 0$ the symmetry is broken and the peaks acquire different heights. This circumstance may be used for experimental definition of the fluctuation correlation in the given system. Note another important effect of the fluctuation correlation on the bistable behavior: at $N_{12} = 0$ both stable states simultaneously lose their stability. The presence of nonzero correlation ($N_{12} \neq 0$) leads to the fact that as the fluctuation power increases, one peak disappears sooner than the other, so that in this case the distribution function goes through the monostable regime before it disappears.

The inequality $C_R \neq C_L$, which is due to the fluctuation correlation, turns out to be quite essential for the limiting ($\varepsilon \to 0$) behavior of the distribution function. As shown by Moss and Welland,[29] in the case where

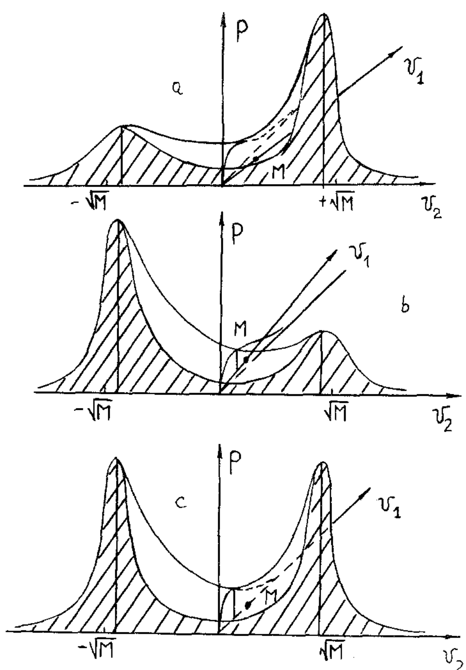

Fig. 2. The distribution function P for the case of $M > 0$. (a) $N_{12} < 0$; (b) $N_{12} > 0$; (c) $N_{12} = 0$.

the U wells are of different depths, the distribution function at the above-mentioned limit transition tends to the δ-function concentrated at the lowest maximum, and only in the case where both wells are of the same depth are the limit distribution functions the sum of the δ-functions concentrated at each maximum. This leads to the fact that in experimental observations of steady states of the above bistable system at very small correlated fluctuations the system must always find itself (at fairly long times of observation) only in one state, which depends on the sign of the correlation coefficient.

5. CONCLUSION

The approximate solution of the two-dimensional stationary Fokker Planck equation obtained in the present work, which corresponds to the stochastic system (II), represents a piecewise-smooth approximation of the stationary distribution function in the nonpotential bistable case. The existence of such smooth approximations was revealed by Graham and Tél,[19,20] who proposed a technique for constructing them on the basis of the method of ε-expansion. This method, however, is inapplicable to systems that are in the vicinity of the stability loss, because of the non-analyticity (to be more exact, nonanalyticity of a nature other than the simple nonanalyticity ε^{-1} of the ε-expansion method) of the solution by bifurcation and noise parameters. The approximate solution obtained is locally Gaussian. But the quadratic form which appears as a local approximation of the system's "potential" is not Taylor terms up to and including the second order of expansion into a Taylor series. This "potential" does not exist near the boundary of the instability region. The closeness of the obtained approximate solution to the true one is determined in the mean (conditional mean for a bistable situation), which just provides the smoothness of the approximation.

The procedure of obtaining equations for moments of (4) and (5) should also be commented upon. For the case of finite M and $\varepsilon \to 0$ the meaning of the values m_i and M_{ij} obtained on their basis as local characteristics of the distribution function was provided, in both monostable and bistable cases, by the Laplace methods. Near the instability boundary, however, as follows from the definition of this boundary, det $\hat{M}$ is close to zero and the asymptotic series in the Laplace method are no longer such. In the monostable case, the above-mentioned method may be looked upon as the replacement of the true distribution function by the closest (in the sense of means and dispersion closeness) Gaussian distribution function. In the bistable case, the situation turns out to be more complicated. The use of the conditional means in obtaining Eqs. (4) and (5) for the calculation of local characteristics of the distribution function, and the replacement of the maximum region integration by the whole space integration, means implicit introduction of characteristic times at which the above actions are valid, namely, if at the initial instant of time the system was in the vicinity of one of the equilibrium states, its corresponding peak of the distribution function is formed faster than the peak near the other equilibrium state, at times smaller than the formation times, and the above treatment is valid.

The use of the local information obtained in this manner for the construction of the global distribution function is equivalent to the assumption that in the course of further evolution toward the stationary state the positions of the distribution function maxima are little affected as compared to those calculated in the first stage of formation of maxima. This assumption looks quite plausible. The same method may be used to avoid such assumptions in constructing smooth approximations of stationary distribution functions near the point of stability loss where the differential properties of the solution are poor and do not permit the calculation of the Taylor expansion of U, but U must be chosen not in the quadratic form, but in the form of the fourth or higher even order. In practice, however, in purely analytical applications, such an approach leads to extremely great computational complications due to the lack of exact expressions for integrals in Eqs. (4) and (5) and the lack of a fairly simple coupling between lower and higher order momenta of the type of (9). Thus, one may hope that in the cases where the rebuilding of the distribution function occurs without merging of maxima at one point, the application of the Gaussian approximation will produce at least correct qualitative results. It is precisely such a situation that takes place for the considered example of two hydrodynamic modes when maxima disappear at different points of the phase space as the point passes in the expanded space of parameters through the instability boundary, i.e., without merging.

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